Policy Gradients

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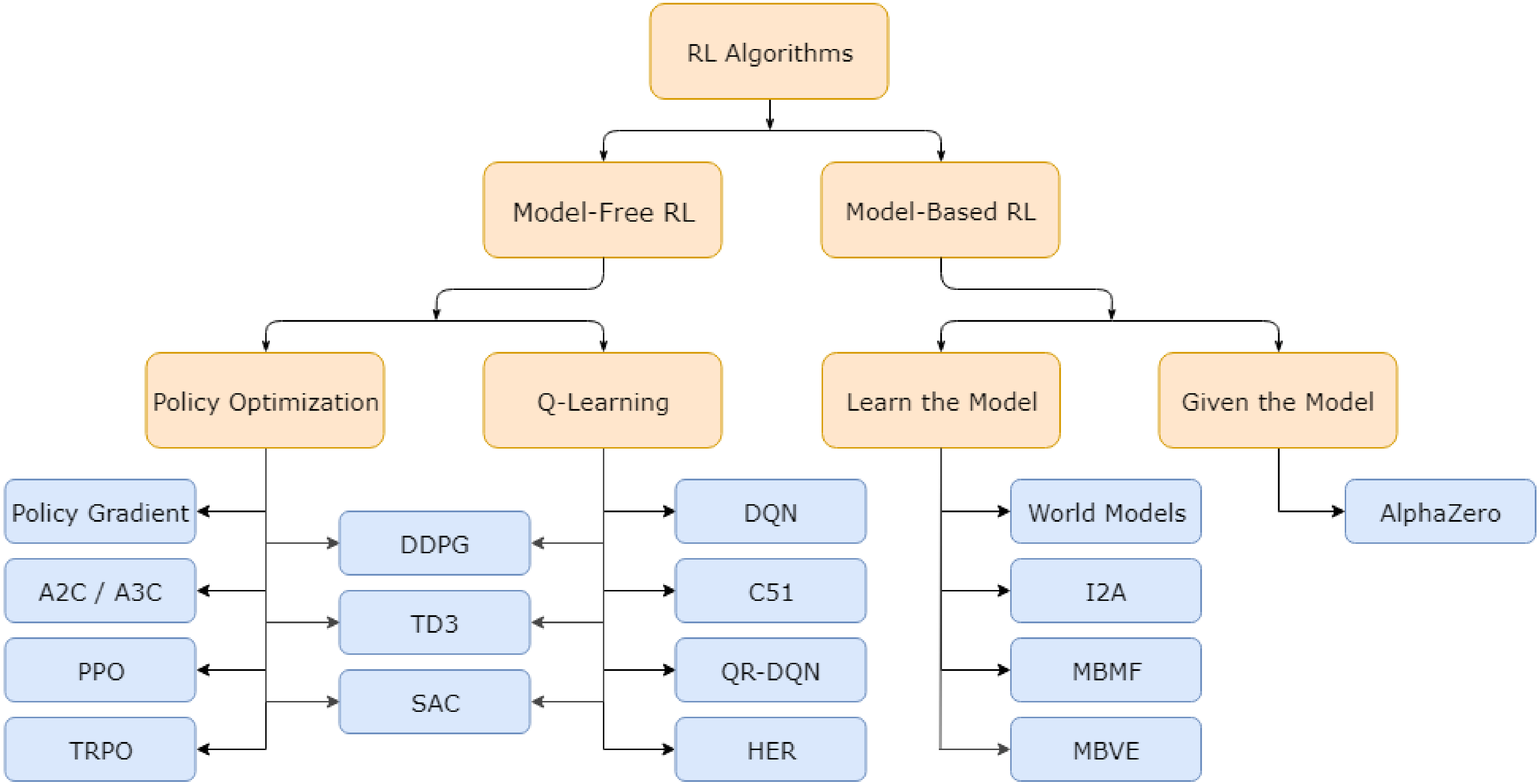
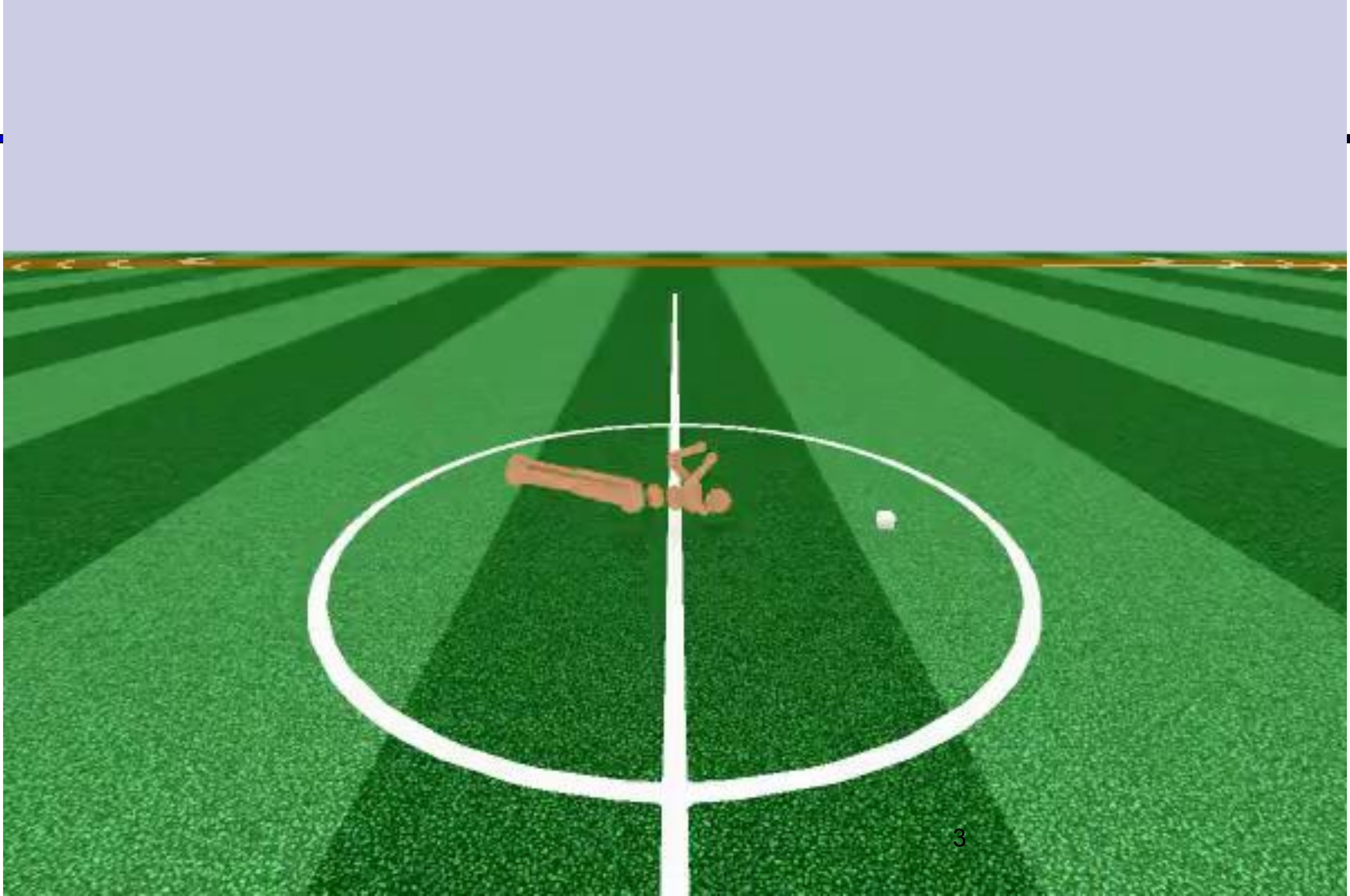
Rough Taxonomy of RL Algorithms





What is the goal of RL?

- Find a policy that maximizes expected utility (discounted cumulative rewards)

$$\pi^* = arg\max_{\pi} E_{\pi}\left[\sum_{t=0}^{\infty} \gamma^t R(s, \pi(s), s')\right]$$

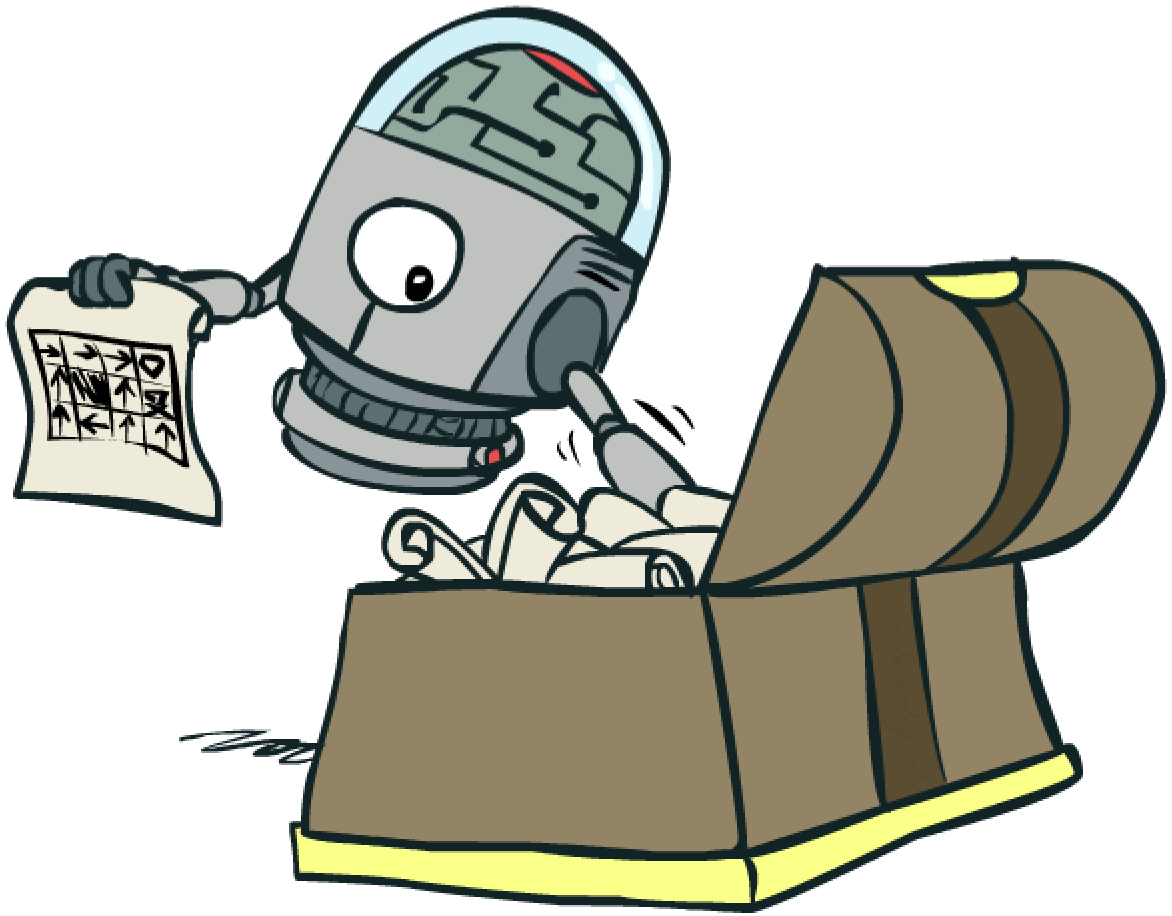
Two approaches to model-free RL

- Learn Q-values
  - Trains Q-values to be consistent. Not directly optimizing for performance.
  - Use an objective based on the Bellman Equation

$$Q_{k+1}(s,a) \leftarrow \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma \max_{a'} Q_k(s',a') \right]$$

- Learn Policy Directly
  - Have a parameterized policy π_θ
  - Update the parameters θ to optimize performance of policy.

Policy Search



Preliminaries

- Trajectory (rollout, episode) $\tau = (s_0, a_0, s_1, a_1, \dots)$
  - $s_0 \sim \rho_0(\cdot)$, $s_{t+1} \sim P(\cdot\,|s_t, a_t)$
- Rewards $r_t = R(s_t, a_t, s_{t+1})$
- Finite-horizon undiscounted return of a trajectory

$$R(\tau) = \sum_{t=0}^{T} r_t$$

- Actions are sampled from a parameterized policy π_θ

$$a_t \sim \pi_\theta(\cdot\,|s_t)$$

Preliminaries

- Probability of a trajectory (rollout, episode) $\tau = (s_0, a_0, s_1, a_1, \dots)$

$$P(\tau|\pi) = \rho_0(s_0) \prod_{t=0}^{T-1} P(s_{t+1}|s_t, a_t)\pi_\theta(a_t|s_t)$$

- Expected Return of a policy $J(\pi)$

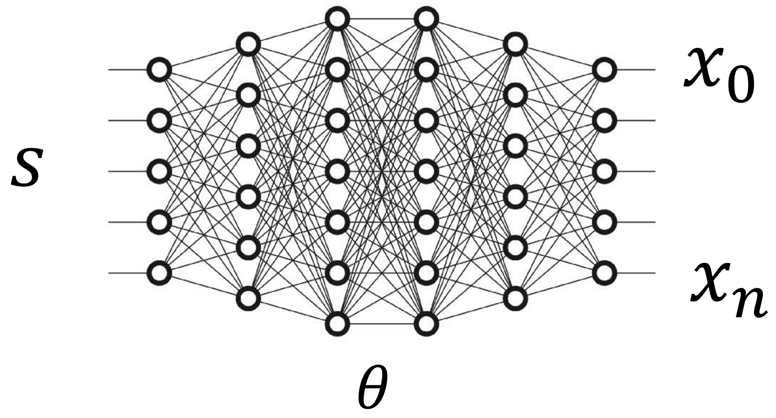
$$J(\pi) = \sum_{\tau} P(\tau|\pi)\, R(\tau) = E_{\tau\sim\pi}[R(\tau)]$$

- Goal of RL: Solve the following optimization problem

$$\pi^* = \underset{\pi}{\operatorname{argmax}}\, J(\pi)$$

How should we parameterize our policy?

- We need to be able to do two things:
  - Sample actions $a_t \sim \pi_\theta(\cdot\,|s_t)$
  - Compute log probabilities $\log \pi_\theta(a_t|s_t)$
- Categorical (classifier over discrete actions)
  - Typically, you output a value x_i for each action (class) and then the probability is given by a softmax equation



$$\pi_\theta(a_i|s) = \frac{\exp(x_i)}{\sum_j \exp(x_j)}$$

How should we parameterize our policy?

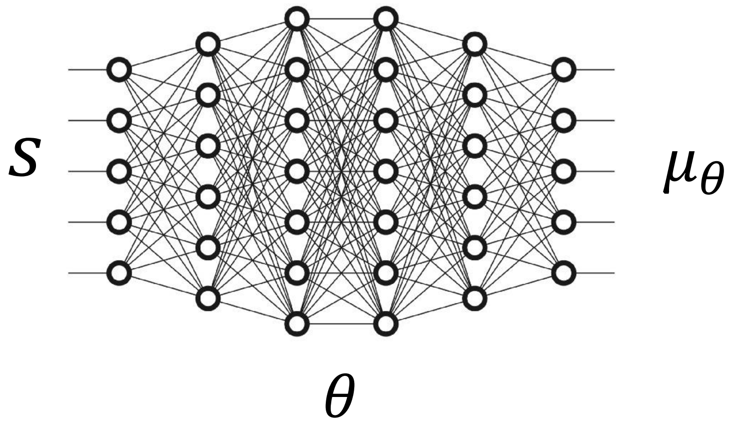
- Diagonal Gaussian (distribution over continuous actions)

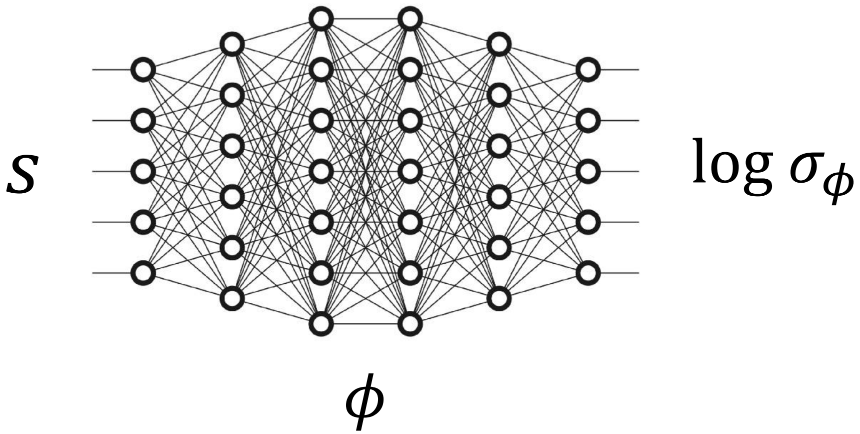
$$a \sim N(\mu, \Sigma)$$

where Σ has non-zero elements only on the diagonal.

Thus, an action can be sampled as

$$a = \mu_\theta(s) + \sigma_\phi(s) \odot z, \qquad z \sim N(0, I)$$





Goal: Update Policy via Gradient Ascent

- We have a parameterized policy and we want to update it so that it maximizes the expected return.
- We want to find the gradient of the return with respect to the policy parameters and step in that direction.

$$\theta_{k+1} \leftarrow \theta_k + \alpha \nabla_\theta J(\pi_\theta)\Big|_{\theta_k}$$

Policy gradient

Fact #1

- Probability of a trajectory:
  - The probability of a trajectory $\tau = (s_0, a_0, \dots s_{T+1})$ given that actions come from π_θ is

$$P(\tau|\pi) = \rho_0(s_0) \prod_{t=0}^{T} P(s_{t+1}|s_t, a_t)\pi_\theta(a_t|s_t)$$

Fact #2

- Log-probability of a trajectory:
  - The log-probability of a trajectory $\tau = (s_0, a_0, \ldots s_{T+1})$ given that actions come from π_θ is

$$\log P(\tau|\pi) = \log \left(\rho_0(s_0) \prod_{t=0}^{T} P(s_{t+1}|s_t, a_t)\pi_\theta(a_t|s_t) \right)$$

$$= \log \rho_0(s_0) + \sum_{t=0}^{T} (\log P(s_{t+1}|s_t, a_t) + \log \pi_\theta(a_t|s_t))$$

Fact #3

- Grad-Log-Prob of a Trajectory
  - Note that gradients of everything that doesn't depend on θ is 0.

$$\nabla_\theta \log P(\tau|\theta) = \nabla_\theta \log \rho_0(s_0) + \sum_{t=0}^{T} (\nabla_\theta \log P(s_{t+1}|s_t, a_t) + \nabla_\theta \log \pi_\theta(a_t|s_t))$$

$$= \sum_{t=0}^{T} (\nabla_\theta \log \pi_\theta(a_t|s_t))$$

Fact #4

- Log-Derivative Trick:
  - This is based on the rule from calculus that the derivative of log x is 1/x

$$\nabla_\theta P(\tau|\pi) = P(\tau|\pi)\nabla_\theta \text{logP}(\tau|\theta)$$

$$\frac{d}{dx}\log g(x) = \frac{1}{g(x)}\frac{d}{dx}g(x) \quad \Rightarrow \quad g(x)\frac{d}{dx}\log g(x) = \frac{d}{dx}g(x)$$

Derivation of Policy Gradient

$$\nabla_\theta J(\pi_\theta) = \nabla_\theta E_{\tau \sim \pi_\theta}[R(\tau)]$$

Try it!

The Policy Gradient (REINFORCE)

- We can now perform gradient ascent to improve our policy!

$$\theta_{k+1} \leftarrow \theta_k + \alpha \nabla_\theta J(\pi_\theta) \Big|_{\theta_k}$$

$$\nabla_\theta J(\pi_\theta) = E_{\tau \sim \pi_\theta} \left[\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t | s_t) \; R(\tau) \right]$$

Estimate with a sample mean over a set D of policy rollouts given current parameters

$$\approx \frac{1}{|D|} \sum_{\tau \in D} \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t | s_t) \; R(\tau)$$

How would you implement this?

1. Start with random policy parameters θ_0
2. Run the policy in the environment to collect N rollouts (episodes) of length T and save returns of each trajectory.

$$a_t \sim \pi_\theta(\cdot\,|s_t) \Rightarrow (s_0, a_0, r_0, s_1, a_1, r_1, \dots, r_T, s_{T+1})$$

$$D = \{\tau_1, \dots \tau_N\}, \qquad R = \{R(\tau_1), \dots R(\tau_N)\}$$

3. Compute policy gradient

$$\nabla_\theta J(\pi_\theta) = E_{\tau\sim\pi_\theta}\left[\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t)\; R(\tau)\right]$$

4. Update policy parameters

$$\theta_{k+1} \leftarrow \theta_k + \alpha \nabla_\theta J(\pi_\theta)\Big|_{\theta_k}$$

5. Repeat from step 2

What does the log probability look like?

- $\log \pi_\theta(a|s) = ?$

$$\pi_\theta(a|s) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(a-\mu)^2}{2\sigma^2}\right)$$

https://en.wikipedia.org/wiki/Normal_distribution

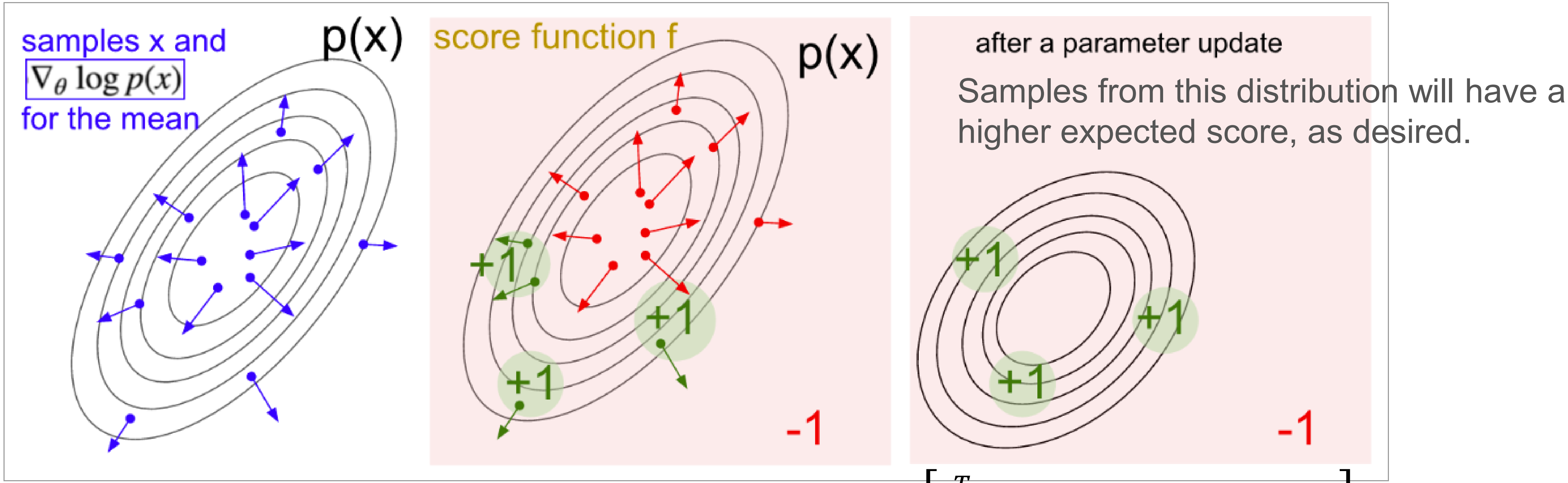
$$\log \pi_\theta(a|s) = -\frac{1}{2}\left[\log(2\pi\sigma^2) + \frac{(a-\mu)^2}{\sigma^2}\right]$$

Some more intuition (thanks to Andrej Karpathy)

- Blue Dots: samples from Gaussian
- Blue arrows: gradients of the log probability with respect to the gaussian's mean parameter

- We score each sample
- Red have score -1
- Green have scores +1

- To update the Gaussian mean parameter, we average up all the green arrows, and the *negative* of the red arrows.



$$\nabla_\theta J(\pi_\theta) = E_{\tau\sim\pi_\theta}\left[\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t)\ R(\tau)\right]$$

https://karpathy.github.io/2016/05/31/rl/

Simple Pytorch Pseudocode

```
for episode in range(num_episodes):
    state = env.reset()
    trajectory = []

    while True:
        action, log_prob = select_action(policy_net, state)
        next_state, reward, done, _ = env.step(action)

        trajectory.append((log_prob, reward))
        state = next_state

        if done:
            break
```

```
# Compute returns and policy loss
log_probs, rewards = zip(*trajectory)
returns = compute_returns(rewards, gamma)
policy_loss = -sum(log_prob * G
                   for log_prob, G in zip(log_probs, returns))

# Update policy network
optimizer.zero_grad()
policy_loss.backward()
optimizer.step()
```

Policy Gradient RL Algorithms

- We can directly update the policy to achieve high reward.
- Pros:
  - Directly optimize what we care about: Utility!
  - Naturally handles continuous action spaces!
  - Can learn specific probabilities for taking actions.
  - Often more stable than value-based methods (e.g. DQN).
- Cons:
  - On-Policy -> Sample-inefficient we need to collect a large set of new trajectories every time the policy parameters change.
  - Q-Learning methods are usually more data efficient since they can reuse data from any policy (Off-Policy) and can update per sample.

Many forms of policy gradients

$$\nabla_\theta J(\pi_\theta) = \underset{\tau \sim \pi_\theta}{\mathrm{E}} \left[\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t) \Phi_t \right]$$

What we derived: $\Phi_t = R(\tau),$

Follows a similar derivation: $\Phi_t = \sum_{t'=t}^{T} R(s_{t'}, a_{t'}, s_{t'+1}),$

https://medium.com/@thechrisyoon/deriving-policy-gradients-and-implementing-reinforce-f887949bd63

- What is better about the second approach?
  - Focuses on rewards in the future!
  - Less variance -> less noisy gradients.

Many forms of policy gradients

$$\nabla_\theta J(\pi_\theta) = \mathop{\mathrm{E}}_{\tau \sim \pi_\theta} \left[\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t) \Phi_t \right]$$

$$\Phi_t = \sum_{t'=t}^{T} R(s_{t'}, a_{t'}, s_{t'+1}),$$

Looks familiar….

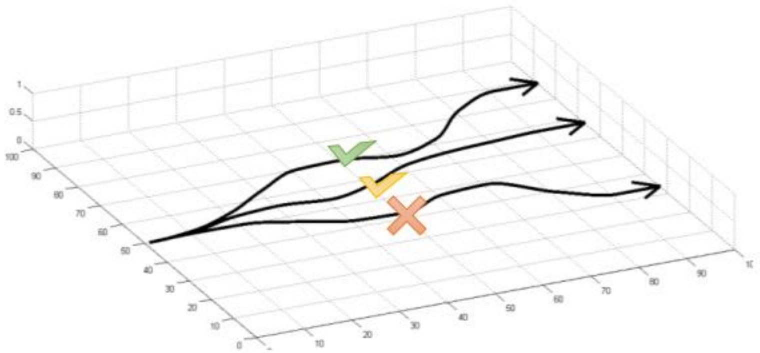
$$\Phi_t = Q^{\pi_\theta}(s_t, a_t)$$

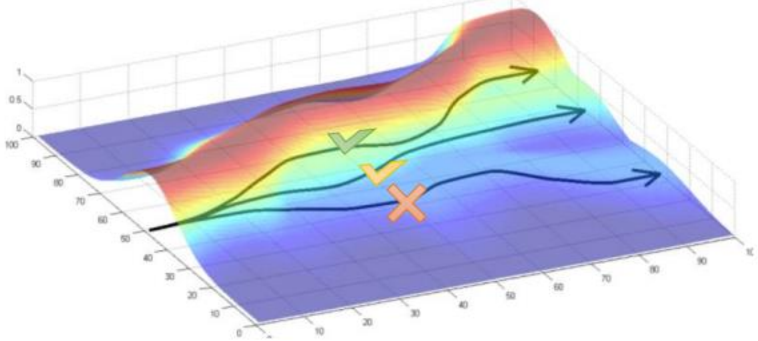
- Now we have an approach that combines a parameterized policy and a parameterized value function!

Baselines

$$\nabla_\theta J(\pi_\theta) = E_{\tau\sim\pi_\theta}\left[\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t)\ R(\tau)\right]$$

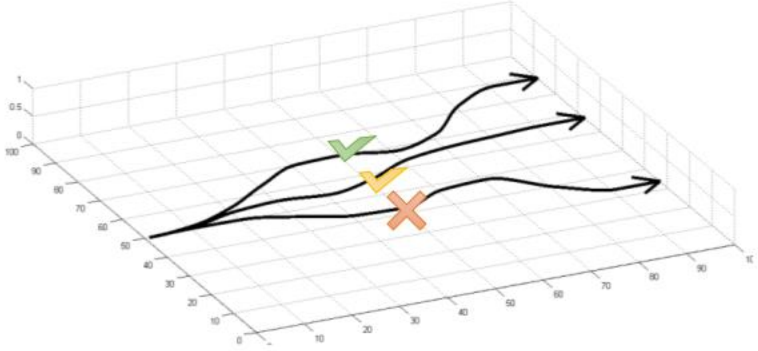
$$\approx \frac{1}{|D|}\sum_{\tau\in D}\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t)\ R(\tau)$$





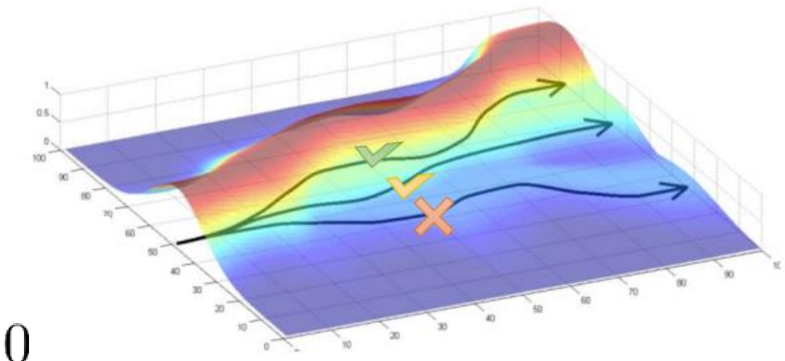
Baselines

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \nabla_\theta \log p_\theta(\tau)[r(\tau) - b]$$



$$b = \frac{1}{N} \sum_{i=1}^{N} r(\tau)$$

But can we do this?



$$E[\nabla_\theta \log p_\theta(\tau) b] = \int p_\theta(\tau) \nabla_\theta \log p_\theta(\tau) b \, d\tau$$

$$= \int \nabla_\theta p_\theta(\tau) b \, d\tau = b \nabla_\theta \int p_\theta(\tau) d\tau = b \nabla_\theta 1 = 0$$

Many forms of policy gradients

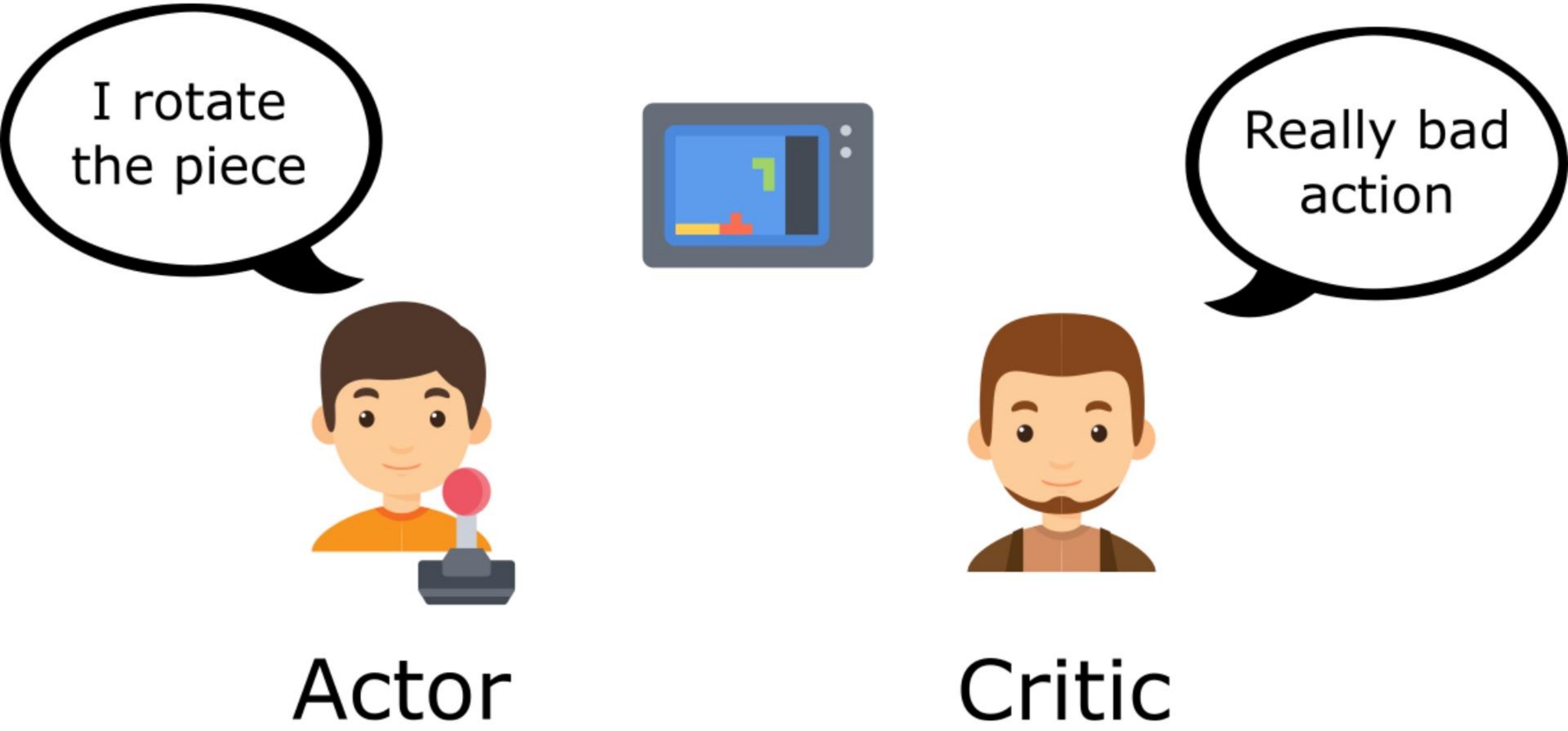
$$\nabla_\theta J(\pi_\theta) = \mathop{\mathrm{E}}_{\tau \sim \pi_\theta} \left[\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t) \Phi_t \right]$$

$$\Phi_t = R(\tau), \qquad \Phi_t = \sum_{t'=t}^{T} R(s_{t'}, a_{t'}, s_{t'+1}), \qquad \Phi_t = Q^{\pi_\theta}(s_t, a_t)$$

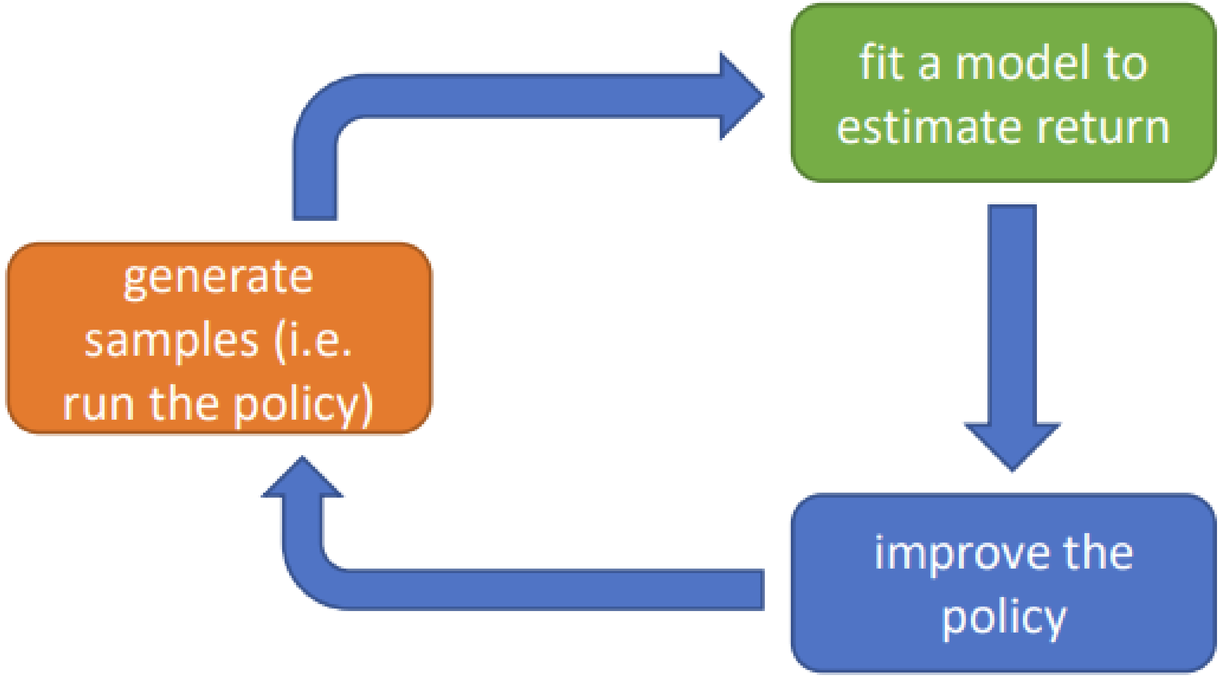
$$\Phi_t = \sum_{t'=t}^{T} R(s_{t'}, a_{t'}, s_{t'+1}) - b(s_t)$$

$$\Phi_t = A^{\pi}(s_t, a_t) = Q^{\pi}(s_t, a_t) - V^{\pi}(s_t)$$

Advantage Function


I rotate the piece
Really bad action
Actor
Critic

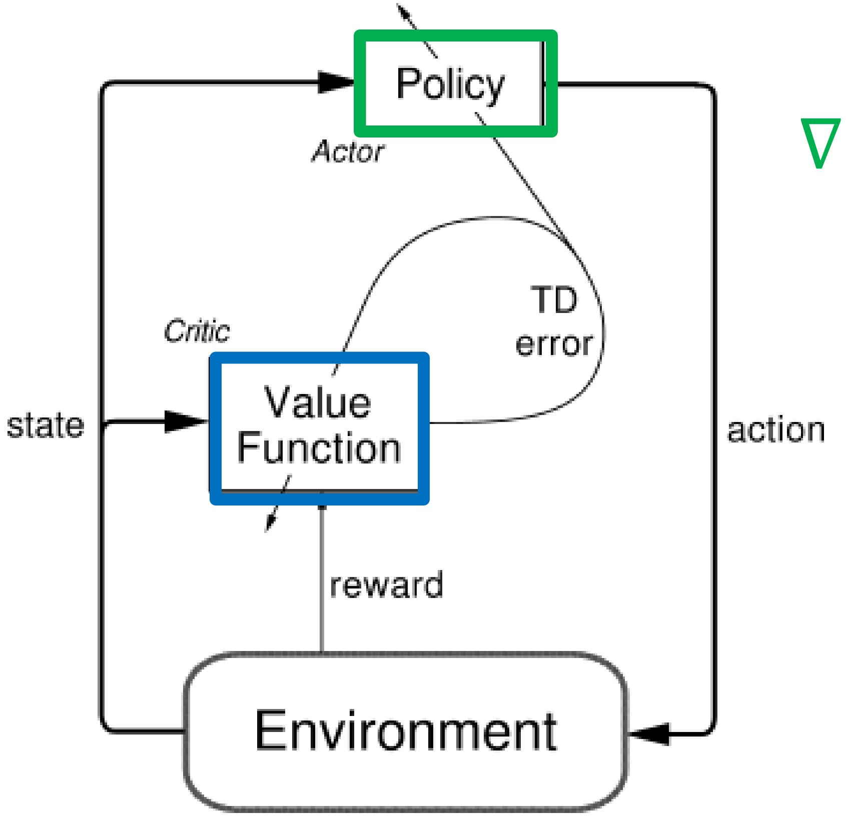
$$\text{fit } \hat{V}^{\pi}_{\phi}$$



$$\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$$

Actor Critic Algorithms

- Combining value learning with direct policy learning
  - One example is policy gradient using the advantage function



$$\nabla_\theta J(\pi_\theta) = E_{\tau\sim\pi_\theta}\left[\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t)\, Q_w^{\pi_\theta}(s_t, a_t)\right]$$

$$\theta_{k+1} \leftarrow \theta_k + \alpha \nabla_\theta J(\pi_\theta)\Big|_{\theta_k}$$

$$\delta = (r_t + \gamma Q_w^{\pi_\theta}(s_{t+1,} a_{t+1}) - Q_w^{\pi_\theta}(s_t, a_t))$$

$$w_{k+1} \leftarrow w_k + \alpha \delta_t \nabla_\theta Q_w^{\pi\_\theta}$$

Q Actor Critic Algorithm Pseudo Code

Algorithm 1 Q Actor Critic

Initialize parameters s, θ, w and learning rates α_θ, α_w; sample $a \sim \pi_\theta(a|s)$.
for $t = 1 \dots T$: **do**
  Sample reward $r_t \sim R(s, a)$ and next state $s' \sim P(s'|s, a)$
  Then sample the next action $a' \sim \pi_\theta(a'|s')$
  Update the policy parameters: $\theta \leftarrow \theta + \alpha_\theta Q_w(s, a) \nabla_\theta \log \pi_\theta(a|s)$; Compute the correction (TD error) for action-value at time t:
    $\delta_t = r_t + \gamma Q_w(s', a') - Q_w(s, a)$
  and use it to update the parameters of Q function:
    $w \leftarrow w + \alpha_w \delta_t \nabla_w Q_w(s, a)$
  Move to a $\leftarrow a'$ and s $\leftarrow s'$
end for

Adapted from Lilian Weng's post "Policy Gradient algorithms"

The Advantage Function

$$A(s,a) = \underbrace{Q(s,a)}_{\text{q value for action a in state s}} - \underbrace{V(s)}_{\text{average value of that state}}$$

q value for action a in state s

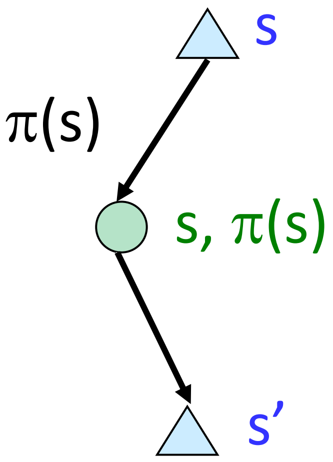
average value of that state

- Why good?

- Why bad?

Temporal Difference Learning

- Big idea: learn from every experience!
  - Update V(s) each time we experience a transition (s, a, s', r)
  - Likely outcomes s' will contribute updates more often



- Temporal difference learning of values
  - Policy still fixed, still doing evaluation!
  - Move values toward value of whatever successor occurs: running average

Sample of V(s): $sample = R(s, \pi(s), s') + \gamma V^{\pi}(s')$

Update to V(s): $V^{\pi}(s) \leftarrow V^{\pi}(s) + \alpha(sample - V^{\pi}(s))$

The Advantage Function

$$A(s,a) = \boxed{Q(s,a)} - V(s)$$

$$r + \gamma V(s')$$

$$A(s,a) = \underbrace{r + \gamma V(s') - V(s)}_{\text{TD Error}}$$

Advantage Actor Critic (A2C)

- Combining value learning with direct policy learning
  - One example is policy gradient using the advantage function

$$\nabla_\theta J(\pi_\theta) = \mathop{\mathrm{E}}_{\tau \sim \pi_\theta}\left[\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t)\Phi_t\right]$$

$$\Phi_t = A^\pi(s_t, a_t) = Q^\pi(s_t, a_t) - V^\pi(s_t)$$

TD error $\delta_t = r(s_t, a_t) + \gamma V^\pi(s_{t+1}) - V^\pi(s_t)$

TD-Learning update

$$w_{k+1} \leftarrow w_k + \alpha \delta_t \nabla_w V(s, a; w)$$

Rough Taxonomy of RL Algorithms